**MATH2048 Honours Linear Algebra II**

**Midterm Examination 2**

**Please show all your steps**, unless otherwise stated. Answer all **five** questions.

1. Let $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ be a linear operator defined by:

$$T(a + bx + cx^2 + dx^3) = (b - d)x + (c + d)x^2 + (d - a)x^3.$$

Find all eigenvalues of $T$, as well as their algebraic multiplicity $\mu_T(\lambda)$ and geometric multiplicity $\gamma_T(\lambda)$. Determine whether $T$ is diagonalizable.

2. Let $V = M_{2\times 2}(\mathbb{R})$ and $T : V \to V$ be the linear operator defined by $T(M) = AMB$, where $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$.

   (a) Find a polynomial $g \in P_3(\mathbb{R})$ such that $T^4 = g(T)$. (**Hint:** The Cayley-Hamilton Theorem.)

   (b) Let $M_0 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ and $W$ be the $T$-cyclic subspace of $V$ generated by $M_0$. Show that $T^2(M_0) = a_0 M_0 + a_1 T(M_0)$ for some scalars $a_0, a_1 \in \mathbb{R}$. Find $\dim(W)$ and the characteristic polynomial of $T|_W$.

3. Let $T = L_A : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear transformation where $A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ and $\lambda \in \mathbb{C} \setminus \{0\}$ is a nonzero complex number.

   (a) Find all 1-dimensional $T$-invariant subspace of $V$.

   (b) Show that $W$ is a 2-dimensional $T$-invariant subspace if and only if $W = \text{span}(\{(x, y, 0)^t, (y, 0, 0)^t\})$ where $x \in \mathbb{C}$ and $y \in \mathbb{C} \setminus \{0\}$.

   (**Hint:** If $W$ is a $T$-invariant subspace of $V$, then $W$ is also $(T - \lambda I)$-invariant.)

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4. (a) Let $A = (a_{ij})_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{C})$, where $a_{ij}$ is the $i$-th row, $j$-th column entry of $A$. Let $\lambda$ be an eigenvalue of $A$. Show that:

$$\lambda \in \bigcup_{1\leq i\leq n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{1\leq j\leq n, j\neq i} |a_{ij}| \right\}.$$

(**Hint:** Let $\mathbf{x} = (x_1, ..., x_n)^t$. Expand $A\mathbf{x} = \lambda\mathbf{x}$. Consider the $i$-th entry.)

(b) Let $V$ be a $n$-dimensional vector space over $\mathbb{C}$, with an ordered basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$. Given that $n \geq 100$. Consider a linear operator $T : V \to V$ defined by:

$$T(\mathbf{v}_1) = a_1\mathbf{v}_1 + b_{11}\mathbf{v}_2 + b_{12}\mathbf{v}_n$$
$$T(\mathbf{v}_n) = a_n\mathbf{v}_n + b_{n1}\mathbf{v}_1 + b_{n2}\mathbf{v}_{n-1}$$
$$T(\mathbf{v}_k) = a_k\mathbf{v}_k + b_{k1}\mathbf{v}_{k+1} + b_{k2}\mathbf{v}_{k-1} \text{ for } k = 2, 3, ..., n-1.$$

Given that $|a_k| > |b_{k1}| + |b_{k2}|$ for all $1 \leq k \leq n$. Using (a), show that all eigenvalues of $T$ are non-zero.

5. Let $T : V \to W$ be a linear transformation between the vector spaces $V$ and $W$ over $\mathbb{C}$. Let $T^* : W^* \to V^*$ be the dual map of $T$ defined by $T^*(g) = g \circ T$. Prove or disprove that $(W/R(T))^*$ is isomorphic to $N(T^*)$. If it is, please construct an isomorphism between $(W/R(T))^*$ and $N(T^*)$. If it is not, please give a rigorous proof. Please explain your answer with details.

**END OF PAPER**